P.	$\log(1+aP^{\beta})$.		Coeff. of Exp.		P.	$\log(1+aP^{\beta})$.		Coeff. of Exp.	
$0^m.0$	0.000000	251	0.0036445	66	$7^m.0$	9.996053	181	0.0037485	48
0 .1	9.999749	142	36511	37	7 .5	9.995872	177	37533	47
0 .2	9.999607	118	36548	31	8 .0	9.995695	174	37580	46
0 .3	9.999489	105	36579	28	8 .5	9.995521	169	37626	45
0 .4	9.999384	96	36607	25	9 .0	9.995352	167	37671	44
0 .5	9.999288	89	36632	24	9 .5	9.995185	164	37715	43
0 .6	9.999199	84	36656	22	10 .0	9.995021	161	37758	43
0 .7	9.999115	80	36678	20	10 .5	9.994860	158	37801	42
0 .8	9.999035	77	36698	21	11 .0	9.994702	156	37843	42
0 .9	9.998958	73	36719	19	11 .5	9.994546	153	37885	40
1 .0	9.998885	71	36738	18	12 .0	9.994393	151	37925	40
1 .1	9.998814	69	36756	19	12 .5	9.994242	149	37965	40
1 .2	9.998745	67	36775	18	13 .0	9.994093	147	38005	39
1 .3	9.998678	65	36793	16	13 .5	9.993946	146	38044	39
1 .4	9.998613	64	36809	17	14 .0	9.993800	143	38083	38
1 .5	9.998549	62	36826	16	14 .5	9.993657	142	38121	38
1 .6	9.998487	61	36842	16	15 .0	9.993515	141	38159	37
1 .7	9.998426	59	36858	16	15 .5	9.993374	138	38196	37
1 .8	9.998367	59	36874	15	16 .0	9.993236	138	38233	36
1 .9	9.998308	57	36889	15	16 .5	9.993098	136	38269	37
2 .0	9.998251	272	36904	73	17 .0	9.992962	134	38306	36
2 .5	9.997979	254	36977	67	17 .5	9.992828	133	38342	35
3 .0	9.997725	240	37044	62	18 .0	9.992695	132	38377	35
3 .5	9.997485	228	37106	60	18 .5	9.992563	131	38412	35
4 .0	9.997257	218	37166	58	19 .0	9.992432	129	38447	35
4 .5	9.997039	210	37224	56	19 .5	9.992303	129	38482	34
5 .0	9.996829	203	37280	53	20 .0	9.992174	127	38516	34
5 .5	9.996626	196	37333	52	20 .5	9.992047	126	38550	34
6 .0	9.996430	191	37385	51	21 .0	9.991921	126	38584	34
6 .5	9.996239	186	37436	49	21 .5	9.991795		38618	

ON ADJUSTMENT FORMULAS.

BY E. L. DE FOREST, M. A.

(*Continued from page* 86.)

THREE separate tests of good adjustment have been proposed by the present writer, for which see *Smithsonian Report* of 1873, p. 333, and a more recent essay in pamphlet form; (*Interpolation and Adjustment of Series*, New Haven, 1876). The first of these tests is, that if we denote by v the residual error, found by subtracting an adjusted term from the correspoding

given one, and denote by ε_1 the mean error of the given term, then forming the values of $(v \div \varepsilon_1)^2$ for all the terms of the series, we ought to have, denoting the whole number of terms by N,

$$\frac{1}{N} \Sigma \left(\frac{v}{\varepsilon_1}\right)^2 = 1 \pm .6745 \sqrt{\frac{2}{N}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

that is to say, the arithmetical mean of all the values of $(v \div \varepsilon_1)^2$ ought to be approximately equal to unity, and ought not to differ from it by more than its probable error, which is

$$.6745 \sqrt{\frac{2}{N}}.$$

The several values of ε_1 are supposed to be determined from the nature of the observations, each given term being the mean result of a number of observations. The values of ε_1 will often vary for different terms of the series, not merely in inverse ratio to the square root of the number of observations made upon each term, but also as some function of the term itself, so that the mean error of one term cannot always be inferred from that of another term solely by comparing the numbers of observations made upon each. An illustration of this is furnished by formula (96), in the *Smithsonian Report* of 1873, p. 334.

There is an even chance that the true series, if we had it, would satisfy the test of good adjustment just described. If, therefore, we find that our adjusted series does satisfy it, we may consider that a good approximation to the truth has been reached, and perhaps as good as the nature of the case will permit. If the mean of $(v \div \varepsilon_1)^2$ falls below the lower limit, that is falls short of unity by more than its probable error, we must infer that the residuals v are too small, and that the series probably has not been smoothed out quite enough, while if it goes beyond the upper limit, so as to exceed unity by more than its probable error, the inference will be that it has been smoothed too much. It thus becomes interesting to see whether we cannot find some rule to guide us in the choice of a formula which will probably make an adjustment satisfying this test. Let us inquire then, what the most probable value of the arithmetical mean of $(v \div \varepsilon_1)^2$ will be, for any given adjustment formula used.

Assuming that all the terms of the given series are of equal weight, let n be a number such that the probable error, or deviation from zero, of the arithmetical mean of any n consecutive values of v shall be equal to the probable error of an adjusted term, so that we have

$$\varepsilon' = \frac{\varepsilon}{\sqrt{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The probable error of any one of these n terms will be approximately

$$\varepsilon = .6745 \sqrt{\left(\frac{\Sigma_n (v^2)}{n-1}\right)}.$$

If we denote by v_1 the true error, or remainder after subtracting the true value of a term from its given or observed value, we shall also have

$$\varepsilon = .6745 \sqrt{\left(\frac{\Sigma_n (v_1^2)}{n}\right)}.$$

Placing these two values of ε equal to each other, and giving to n its value from (10), we get

$$\frac{\Sigma_n (v^2)}{\Sigma_n (v_1^2)} = 1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2.$$

But the most probable ratio of the two sums in the first member will evidently be unchanged if we extend the summation throughout the whole series of N terms, and dividing both numerator and denominator by ε_1^2, and also by N, we shall have

$$\frac{1}{N} \Sigma\left(\frac{v}{\varepsilon_1}\right)^2 \div \frac{1}{N} \Sigma\left(\frac{v_1}{\varepsilon_1}\right)^2 = 1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2.$$

Now in the first member the most probable value of the denominator is unity, because every value of $v_1 \div \varepsilon_1$ is a true error in a system whose mean error is unity, and the square of the mean error is the mean of the squares of all the true erorrs. Hence we have, as the most probable value of the arithmetical mean of $(v \div \varepsilon_1)^2$

$$\frac{1}{N} \Sigma\left(\frac{v}{\varepsilon_1}\right)^2 = 1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It thus appears that this mean value will most probably be less than unity by the square of the error-ratio due to the adjustment formula employed, though it may and probably will exceed or fall short of this value in particular cases.* We have reached this result by assuming that the weights of the given terms are all equal, yet it will hold true approximately if they are unequal, for the probable value of v will vary in nearly the same ratio as ε_1 does.

Since the proposed test of good adjustment requires that the mean of $(v \div \varepsilon_1)^2$ shall not fall short of unity by more than its probable error, which is

$$.6745 \sqrt{\frac{2}{N}}$$

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*Its probable error, or probable deviation from the most probable value, will be

$$\pm .6745 \left\{ 1 - \left[\frac{\varepsilon'}{\varepsilon}\right]^2 \right\} \sqrt{\frac{2}{N}}.$$

Compare *Interpolation and Adjustment of Series*, p. 10.

if the adjusted series is the true one, it follows that we ought to choose an adjustment formula whose error-ratio $\varepsilon' \div \varepsilon$ is at any rate small enough to satisfy the equation

$$1 - \left(\frac{\varepsilon'}{\varepsilon}\right)^2 = 1 - .6745\sqrt{\frac{2}{N}}$$

that is, we should have

$$\frac{\varepsilon'}{\varepsilon} \leqq \sqrt{\left(.6745\sqrt{\frac{2}{N}}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which may be written

$$\frac{\varepsilon'}{\varepsilon} \leqq \frac{.9767}{\sqrt[4]{N}}.$$

If we wish to adjust a series of 50 terms, we cannot consider that the chances are in favor of the test being satisfied, unless we use an adjustment formula whose error-ratio does not exceed

$$\frac{.9767}{\sqrt[4]{50}} = .367,$$

so that we should choose a formula of at least as many as 17 terms in Table A., or 23 in Table B., or 19 in Table C. On the other hand, formula (12) gives

$$N \leqq 2(.6745)^2 \div \left(\frac{\varepsilon'}{\varepsilon}\right)^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

that is to say $\qquad N \leqq \dfrac{.9099}{(\varepsilon' \div \varepsilon)^4}.$

Assigning to $\varepsilon' \div \varepsilon$ the successive values .300, .345, .311,
we find for N the values 112, 64, 97,
showing that while a series of as many as 112 terms can probably be adjusted by the 25-term formula of Table A. so as to meet the test, the longest formula of Table B. would probably succeed only for a series of 64 terms or less, and that of Table C. for one of 97 terms or less.

The conclusion we come to from this investigation is, that it will be advisable in most cases to use the longest adjustment formulas given in the tables, provided that this can be done without violating the condition that any $2m+1$ included terms must not deviate greatly from the form of a series of the third or any lower order.

As the first m and last m terms of the series cannot be reached directly by the formula, the series should be graphically extended by m terms at both ends, first plotting the observations on paper as ordinates, and then extending the curve along what seems to be its probable course, and measuring the ordinates of the extended portions. It is not necessary that this extension should coincide with what would be the true course of the curve in those parts.

The important point is, that the m terms thus added, taken together with the $m+1$ adjacent given terms, should follow a curve whose form is approximately algebraic and of a degree not higher than the third. As the adjusting process requires repeated multiplications by the coefficients l, it will be well to prepare in advance a table showing the product of each of these coefficients by each of the nine digits. It will also answer every purpose, and save labor, if we adjust only alternate terms in the regular way, and then fill in the intermediate terms from these by the simple formula for "interpolation into the middle", which may be written

$$u_0 = \tfrac{1}{16}[9(u_1+u_{-1})-(u_3+u_{-3})]. \quad \ldots\ldots \quad (14)$$

Two additional tests of good adjustment have been discussed by the writer, one of which depends upon the fortuitous grouping of the + and — signs in the series of residual errors v. (*Interpolation and Adjustment of Series*, p. 31.) This is much the simplest test of any, its use requiring little labor or knowledge of mathematics. It was shown that in the case of a periodic series, that is, a series whose first and last terms are consecutive, the most probable number of isolated groups of n like signs, occurring from accidental causes, will be

$$\frac{N}{2^{n+1}} \pm \frac{.6745}{2^{n+1}}\sqrt{(2^{n+1}-1)N} \quad \ldots\ldots\ldots \quad (15)$$

where the expression which follows the doubtful sign is the probable error. This will apply sufficiently well to the case of ordinary or non-periodic series, when the method of adjustment here discussed has been used, provided that we treat the first and last signs of v as consecutive, so that if they are alike, they belong to the same group. Taking $n=1$, we find that the most probable number of isolated single signs, that is, signs unlike both the adjacent ones, is

$$\tfrac{1}{4}N \pm \tfrac{.6745}{4}\sqrt{(3N)}, \quad \ldots\ldots\ldots \quad (16)$$

and for $n = 2$, the most probable number of groups of two like signs is found to be

$$\tfrac{1}{8}N \pm \tfrac{.6745}{8}\sqrt{(7N)},$$

so that the most probable number of signs falling within groups of two is

$$\tfrac{1}{4}N \pm \tfrac{.6745}{4}\sqrt{(7N)}. \quad \ldots\ldots\ldots \quad (17)$$

Adding (16) and (17) together, we have for the most probable number of signs falling within isolated groups of either one or two like signs

$$\tfrac{1}{2}N \pm \tfrac{.6745}{4}\sqrt{(10N)}, \quad \ldots\ldots\ldots \quad (18)$$

which may be written

$$\tfrac{1}{2}N \pm .533\sqrt{N},$$

and this is also the expression for the most probable number of signs falling in groups of more than two. Hence we have this practical rule, that if a series has been well adjusted, the whole number of signs of the residual v which fall within groups of only one or two like signs each, will probably be about equal to the whole number which fall within groups of more than two, and the probable error of either number is

$$0.533\sqrt{N}.$$

If the number of signs in groups of one or two should exceed $\frac{1}{2}N$ by more than this probable error, the inference will be that the series probably has not been smoothed out enough, whereas if they fall short of $\frac{1}{2}N$ by more than the probable error, we must presume that the series has been smoothed too much.

It was also shown in the treatise referred to, p. 33, that if in formula (15) the most probable number $N \div 2^{n+1}$ is found to be less than $\frac{1}{4}$ for any assumed value of n, it will show that the odds are against the occurrence of any group of so many as n like signs. If we take

$$\frac{1}{4} = \frac{N}{2^{n+1}}$$

it gives

$$n = 1 + \frac{\log N}{\log 2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

that is to say

$$n = 1 + 3.32 \log N.$$

Hence we have the rule, that there will be a preponderance of chances against the occurrence, from accidental causes, of any group of a number of like signs greater than

$$1 + 3.32 \log N,$$

and if a larger group does occur, it will indicate a probability that the series has been smoothed out too much, or that its true law, in that vicinity, cannot be fairly represented by an algebraic curve of the third degree, for so many consecutive terms as are included by the adjustment formula.

The writer will take this opportunity to make a remark about the method of constructing equations of curves representing annual variations of temperature, from the monthly means taken as data, referred to at p. 314 of the *Smithsonian Report* of 1871. It was a means of suggesting to my mind a general method of interpolation, and has been referred to as the discovery of Professor Everett, who published it as such in the *Edinburgh New Philosophical Journal* for July 1861, and again in the *American Journal of Science and Arts* for January 1863. I have since learned that the method had

been published some eleven years earlier, by M. Bravais, a French meteorologist. It may be found at p. 324 of Vol. II. of the *Meteorologie* which forms part of the series of the *Voyages de la Commission Scientifique du Nord*, edited for the French Government by M. Gaimard and others. The volume mentioned has no date, but one of the meteorological charts in the accompanying *Atlas de Physique* bears the date 1850.* The analytical process in question, therefore, ought to be designated after its first discoverer, as the method of Bravais, rather than as Everett's method. Of course this earlier origin of the method increases the probability that Schiaparelli was not unacquainted with it, as suggested by me in *Interpolation and Adjustment of Series*, p. 39. The same property has recently been published by a writer in the *Astronomische Nachrichten* for Nov. 24, 1873, and this is catalogued, apparently as a new discovery, in the *Jahrbuch uber die Fortschritte der Mathematik* for that year, p. 123.

An inaccurate remark was made by me at p. 310 of the *Sm. Report* of 1871, in describing M. Tchebychef's mode of arranging data as intended for making "ordinary interpolations, not from groups, but from single terms or ordinates." From a brief allusion to the method in Bertrand's *Differential Calculus* I had supposed that it was something like what is known as the method of normal places. That this was an error, has been shown by a recent examination of Tchebychef's original memoir, which is entitled *Sur l'Interpolation dans le cas d'un grand nombre de donnẽes fournies par les Observations.* The method regards the series of observations as geometrically represented by ordinates, and the area of the polygon formed by joining their extremities is supposed to be divided by certain limiting ordinates into a number of areas, which are regarded as values of the area $\int y dx$ for the required curve, between those limits.† These areas are taken as data for determining the valuos of the constants in the algebraic equation of the curve,

$$y = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n.$$

The constant A_0 is determined from one set of areas, A_1 from another, and A_2 from still another, and so on, the limits of these areas being chosen with a view to securing the best values for the interpolated terms denoted by y. For further details, the reader should consult the original work, in the *Mémoires de l'Académie de Saint Pétersbourg*, 1859.

*The observations were taken in the years 1838—40, and the results were published in a long series of *livraisons*, both of the text and the charts, at different times for many years after the return of the expedition.

†In case observations are equidistant and very near together, each area is regarded as equal to the sum of all the observations falling within its limits, multiplied by the constant interval between two consecutive observations.